Adaptive Filtering Using Constrained Subband Updates

Kong A. Lee and Woon S. Gan
Digital Signal Processing Lab, School of Electrical & Electronic Engineering,
Nanyang Technological University, Singapore 639798.
Email: {ekalee, ewsgan@ntu.edu.sg}

Abstract—This paper describes and analyzes a class of subband adaptive filters (SAFs) stems from various approaches of applying subband and multirate techniques in adaptive filtering. This class of SAFs has a unique structure of weight-control mechanism where subband error signals are used to actuate the adaptation of a fullband weight vector. In this paper, we show that the recursive weight adaptation can be expressed as an underdetermined least-squares solution. Within this deterministic framework, we show its underlying affinity with the normalized least-mean-square (NLMS) and affine projection (AP) algorithms. We then compare the efficacy of these algorithms in terms of convergence performance and computational complexity.

I. INTRODUCTION

Among various adaptive filtering algorithms, the least-mean-square (LMS) algorithm and its normalized version (NLMS) are the most popular and widely used because of their simplicity and robustness characteristics [1]. However, they suffer from slow convergence when the input signal spectrum has a large spectral dynamic range. The convergence problem worsens when the system to be identified involves long impulse response. For example, in acoustic echo cancellation application, adaptive filter with more than one thousand taps has to be used for modeling the acoustics of a room.

Subband and multirate techniques have been employed in designing computationally efficient adaptive filter with improved convergence performance [1]–[6]. In conventional SAF each subband adapts a separate subfilter in its own adaptation loop [2], [3]. The conventional structure can achieve some computational savings, however, detail analysis shows that its convergence performance is limited by aliasing and band-edge effects [2], [3], [7]. To mitigate these structural problems, a new structure of weight-control mechanism has been adopted in [4]–[6]. In the new structure, the modeling filter is no longer separated into subfilters for each of the spectral bands. Conversely, subband signals are used to actuate the adaptation of the fullband tap weights of the modeling filter. This structure of weight-control mechanism is different from that in the conventional SAF and it is a preferred option when the major objective is to obtain a better convergence performance than that of the classical LMS filter [4]–[6].

In [6], the authors propose a multiple-constraint optimization criterion that results in a SAF incorporated with the newly structured weight-control mechanism described above. The adaptive filtering algorithm so devised is referred to as normalized-SAF (NSAF) algorithm. In this paper, we show that the NSAF algorithm can also be derived as a solution to an underdetermined least-squares estimation problem. This approach embodies the algorithm in a deterministic framework enabling us to establish connections between the NSAF, NLMS, and AP algorithms. We then compare various aspects of these algorithms in terms of their design criterion, decorrelating properties, convergence performance, and computational complexity. Our analysis reveals the underlying affinity between these algorithms which facilitates and provides motivation for further analysis and study.

The paper is organized as follows. In Section II, we summarize various forms of SAF presented in [4]–[6]. In Section III, the NSAF algorithm is reformulated as an underdetermined least-squares solution. We then compare the NSAF, NLMS, and AP algorithms under the same formulation. Section IV investigates the inherent decorrelating properties of the NSAF algorithm. In Section V, convergence performance of these algorithms is analyzed and compared through simulations. Finally, Section VI concludes the paper.

II. NEW STRUCTURE OF WEIGHT-CONTROL MECHANISM FOR SAF

The SAFs reported in [4]–[6] are difference approaches to and embellishments of the basic form depicted in Fig. 1. The input signal \( u(n) \) and desired response \( d(n) \) are partitioned into \( N \) subbands by mean of analysis filters \( H_0(z), \ldots, H_{N-1}(z) \). Each subband signal occupies a portion of the original frequency band. Hence, the outputs of the \( N \)-input \( N \)-output system \( W(z)I_{N\times N} \), i.e., \( y_i(n) = \sum_{m=0}^{M-1} w_{im} u_i(n-1) \) for \( i = 0,1,\ldots,N-1 \), are essentially the responses of the transversal filter \( W(z) = \sum_{m=0}^{M-1} w_m z^{-m} \) to the input signal \( u(n) \) in each of the \( N \) contiguous spectral bands. It should be noted that \( W(z)I_{N\times N} \) is an uncoupled system and it is basically a bank of parallel filters with identical transfer function \( W(z) \) of order \( M - 1 \).
To preserve the total effective sampling rate \( N \) subbands at \( 1/N \text{th} \) the original sampling rate), the subband filter outputs and desired responses are critically decimated, i.e., \( y_{i,D}(k) = y_i(kN) = w^T u_i(k) \) and \( d_{i,D}(k) = d_i(kN) \), where

\[
u_i(k) = [u_i(kN), u_i(kN-1), \ldots, u_i(kN-M+1)]^T \tag{1}\]

is the input data vector for the \( i \)th subband, and the vector \( w = [w_0, w_1, \ldots, w_{M-1}]^T \) holds the coefficients of the transversal filter \( W(z) \). The superscript \( T \) denotes matrix transposition. The difference between the subband signals, \( e_{i,D}(k) = d_{i,D}(k) - w^T u_i(k) \), measures how far the filter output is from the desired response, in each of the \( N \) spectral bands at the decimated rate. The estimation errors in all the \( N \) subbands, \( e_D(k) = [e_{0,D}(k), e_{1,D}(k), \ldots, e_{N-1,D}(k)]^T \), can be represented in a more compact form as

\[
e_D(k) = d_D(k) - U^T(k) w, \tag{2}\]

where we have defined the data matrix \( U(k) \) and desired response vector \( d_D(k) \), respectively, as

\[
U(k) = [u_0(k), u_1(k), \ldots, u_{N-1}(k)] \quad \text{and} \quad d_D(k) = [d_0(k), d_1(k), \ldots, d_{N-1}(k)]^T. \tag{3}\]

II. UNDERDETERMINED LEAST-SQUARES SOLUTIONS

In this section, we show that the weight recursion obtained in [6] is indeed the minimum-norm solution to an underdetermined least-squares estimation problem. Within this deterministic framework, the NSAF algorithm can be viewed as a generalization of the NLMS algorithm or alternatively as a subband variant of the AP algorithm.

A. Constrained Subbands Updates

Consider the situation where a fullband weight vector \( w(k) \) is to be updated into a presumably better estimate \( w(k+1) \) of an unknown system \( \w^* \) using the accessible subband data \( \{U(k), d_0(k), e_0(k)\} \). A good criterion that would ensure convergence to the optimum solution \( \w^* \), after sufficient number of iterations, is to have the updated weight vector \( w(k+1) \) nulls the a posteriori estimation errors

\[
e_D(k) = d_D(k) - U^T(k) w(k) \tag{6}\]

on the updated weight vector \( w(k+1) \). Knowing that the a priori estimation errors are given by \( e_0(k) = d_0(k) - U^T(k) w(k) \), the set of \( N \) constraints (6) is equivalent to

\[
U^T(k) \delta w(k+1) = e_0(k), \tag{7}\]

where \( \delta w(k+1) = w(k+1) - w(k) \) denotes the weight adjustment. Equation (7) defines an underdetermined case of least-squares estimation problem where the number of parameters \( M \) is more than the number of constraints \( N \) (i.e., the number equations in the linear system). Assuming that \( U^T(k) \) has full row rank, there exist an infinite number of solutions to this underdetermined system. However, there is one solution that gives a minimum perturbation from \( w(k) \) to \( w(k+1) \):

\[
\delta w(k+1) = U(k)[U^T(k)U(k)]^{-1} e_0(k), \tag{8}\]

where \( U(k)[U^T(k)U(k)]^{-1} \) is the pseudoinverse of \( U^T(k) \). This special value of \( \delta w(k+1) \) is termed as the minimum-norm solution in the literature. The minimum perturbation requirement is justified by the principle of minimal disturbance [1, pp. 321] which states that the updated weight vector \( w(k+1) \) should be as close as possible to the previous weight vector \( w(k) \) subject to a set of hard constraint(s) imposed on the updated filter output(s). Here, the set of hard constraints is defined in (6) or its equivalent form in (7).
is introduced

\[ \delta w(n+1) = u(n)[u^T(n)u(n)]^{-1}e(n), \]

where

\[ e(n) = d(n) - u^T(n)w(n), \]

and

\[ u(n) = [u(n), u(n-1), ..., u(n-M+1)]^T. \]

AP

\[ \delta w(n+1) = A(n)[A^T(n)A(n)]^{-1}e(n), \]

where

\[ A(n) = [u(n), u(n-1), ..., u(n-N+1)], \]

\[ e(n) = d(n) - A^T(n)w(n), \]

and

\[ d(n) = [d(n), d(n-1), ..., d(n-N+1)]^T. \]

TABLE I. DETERMINISTIC INTERPRETATION OF THE NLMS AND AP ALGORITHMS

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Underdetermined least-squares solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLMS</td>
<td>[ \delta w(n+1) = u(n)[u^T(n)u(n)]^{-1}e(n), ] where [ e(n) = d(n) - u^T(n)w(n), ] and [ u(n) = [u(n), u(n-1), ..., u(n-M+1)]^T. ]</td>
</tr>
<tr>
<td>AP</td>
<td>[ \delta w(n+1) = A(n)[A^T(n)A(n)]^{-1}e(n), ] where [ A(n) = [u(n), u(n-1), ..., u(n-N+1)], ] [ e(n) = d(n) - A^T(n)w(n), ] and [ d(n) = [d(n), d(n-1), ..., d(n-N+1)]^T. ]</td>
</tr>
</tbody>
</table>

Finally, by exploiting subband-orthogonality properties of the filter bank which leads to the diagonal assumption [6], the weight adaptation equation (8) reduces to a simple form:

\[ w(k+1) = w(k) + \mu \sum_{i=0}^{N-1} u_i(k)\left[u^T_i(k)u_i(k) + \alpha \right]^{-1}e_i(k). \] (9)

A positive step-size parameter denoted by \( \mu \) is introduced in the recursive relation to exercise control over the change in the weight vector, and \( \alpha \) is a small positive constant to avoid numerical difficulties when the input level is too low.

B. Generalized Form of the NLMS Algorithm

It has been shown in [1], [8] that the NLMS and AP algorithms can also be derived as underdetermined least-squares solutions as listed in Table I. Clearly, an attribute that NLMS, AP, and NSAF algorithms have in common is that they are manifestation of the principle of minimal disturbance which requires that the updated weight vector to be as close as possible to the previous weight vector subject to a single fullband constraint (for the case of NLMS algorithm), or multiple constraints (for the case of AP and NSAF algorithms).

Compare the minimum-norm solutions in (8) and Table I, both AP and NSAF algorithms can be seen as generalized forms of NLMS algorithm. The AP algorithm generalizes the NLMS along the time-axis using multiple time-domain constraints [1], [8], whereas the NSAF algorithm generalizes the NLMS along the frequency-axis using multiple subband constraints. In particular, for the special case of \( N=1 \), the NSAF algorithm (9) reduces to the NLMS algorithm

\[ w(k+1) = w(k) + \mu u(k)\left[u^T(k)u(k) + \alpha \right]^{-1}e(k). \] (10)

In this case, the filter bank reduces to a single filter with unit impulse response \( h(n) = \delta(n) \) because band-partitioning and the subsequent signal reconstruction are no longer required. Since \( k = n \) for \( N=1 \), the weight adaptation (10) is performed at the original sampling rate.

C. Geometric Interpretation

From a geometric interpretation [9], the NSAF algorithm can be seen as a subband variant of the AP algorithm where subband basis \( u_i(k), \ldots, u_{N-1}(k) \) is used in place of the time-domain basis \( u(n), \ldots, u(n-N+1) \) (i.e., delayed versions of the tap-input vector) in the iterative projection. The major difference is on the nature of the basis used in the projection. The advantage of the subband basis is due to its orthogonality properties [6], i.e., the inner product \( u^T_i(k)u_i(k) \) is diagonal, which greatly reduces the computational burden in getting its inverse \( [u^T_i(k)u_i(k)]^{-1} \) for the weight adaptation. The subband basis can be seen as a remedy to the matrix inversion problem encountered in AP algorithm. It can be regarded as a viable alternative to the fast implementation of AP algorithm (FAP). FAP recursively calculates the matrix inversion in which numerical instabilities are introduced, thus requires proper regularization and reinitialization strategy.

IV. MEAN BEHAVIOR

The major motivation of generalizing the NLMS with multiple constraints is to obtain an improved convergence performance. The effectiveness of the NSAF algorithm in dealing with colored excitation can be noticed from the mean behavior of the projection matrix.

Subtracting (8) from \( w^o \), rearranging terms, taking the expectation of both sides, and making the independence assumption [1], we can describe the mean behavior of the weight-error vector \( \varepsilon(k) = w^o - w(k) \) as

\[ E[\varepsilon(k+1)] = [I - \mu R_s]E[\varepsilon(k)], \] (11)

where

\[ R_s = E\{P(k)\} = \sum_{i=0}^{N-1} E\{u_i(k)u_i^T(k)\} \frac{1}{M \gamma_i(0)} \] (12)

is the mean of the projection matrix defined as

\[ P(k) = U(k)[U^T(k)U(k)]^{-1}U^T(k). \] (13)

Here, we have assumed that the input signal \( u(n) \) is wide-sense stationary (WSS), and \( M \) is large enough such that the inner product \( u_i^T(k)u_i(k) \) can be approximated by a constant which is equal to \( M \) times the variance of the subband signal, i.e., \( u_i^T(k)u_i(k) = M \gamma_i(0) \). Clearly, the mean of the projection matrix is a weighted sum of \( N \) subband correlation matrices \( R_s = E\{u_i(k)u_i^T(k)\} \). The matrix \( R_s \) can be completely defined by a weighted spectrum

\[ \Gamma_u(e^{i\omega}) = \frac{1}{M} \sum_{i=0}^{N-1} \frac{1}{\gamma_i(0)} |H_i(e^{i\omega})|^2 \Gamma_y(e^{i\omega}), \] (14)

where \( |H_i(e^{i\omega})| \) is the magnitude response of the \( i \)th analysis filter and \( \Gamma_y(e^{i\omega}) \) is the input power spectrum. Equation (14) indicates that the subband decomposition and normalization operations essentially form a decorrelation filter.
that flattens the spectrum of the input signal. This decorrelating properties of the NSAF algorithm renders its convergence less sensitive to the coloring of the input signal.

V. SIMULATIONS

We consider a system identification problem in the following simulations. The system to be identified $w^*$ is an acoustic response of a room with 300 ms reverberation time and truncated to 1024 taps. The adaptive weight vector has identical length of $M = 1024$. The excitation signal to the adaptive identification system is an AR(2) random process with coefficients $(1.0, -1.6, 0.81)$. White noise is added to the output of the unknown system giving a 40 dB SNR.

The NSAF algorithm is chosen to have $N = 4$ subbands and filter length $L = 32$ for its pseudo-QMF cosine-modulated filter banks. Two variations of sample- and block-based AP algorithm are considered, both with an order of $N = 4$. The block-based AP algorithm is commonly known as partial rank algorithm (PRA) [8]. Step-sizes are chosen such that identical steady-state mean-square error (MSE) is achieved for all the algorithms in order to allow a fair comparison of their transient behavior. Fig. 2 shows the MSE learning curves obtained by ensemble averaging over 200 independent trials. It can be noted that the convergence rate of the NSAF algorithm is close to that of the AP algorithm and faster than that of the PRA. The NSAF algorithm, being of $O(M + NL)$ complexity, is computationally more efficient than the AP and PRA which have complexity $O(N^3M)$ and $O(NM)$, respectively.

Both NSAF and AP algorithms converge faster than NLMS algorithm, mainly due to their inherent decorrelating properties. In particular, the subband decomposition and normalization features of the NSAF form a decorrelation filter that effectively flattens the input spectrum. Fig. 3 shows the power spectrum of the AR(2) process $\Gamma_u(e^{j\omega})$, the corresponding decorrelation filter $H_u(e^{j\omega})$, and the resultant power spectrum $H_u(e^{j\omega}) \Gamma_u(e^{j\omega})$ with a reduced dynamic range. The NSAF algorithm attains a faster convergence than the NLMS with almost equivalent number of multiplications per sampling period ($M \gg NL$ in this case).

VI. CONCLUSIONS

Both NSAF and AP algorithms generalize the NLMS algorithm with multiple subband and time-domain constraints, respectively, giving a significant improvement in convergence rate. The NSAF algorithm can achieve almost equivalent convergence performance as the AP algorithm with computational complexity close to that of the NLMS algorithm. Thus, the NSAF algorithm would be preferable from these aspects, especially for very-high-order adaptive filters.

REFERENCES